Vortex in a weakly relativistic Bose gas at zero temperature and relativistic fluid approximation

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Abstract

The Bogoliubov procedure in quantum field theory is used to describe a relativistic almost ideal Bose gas at zero temperature. Special attention is given to the study of a vortex. The radius of the vortex in the field description is compared to that obtained in the relativistic fluid approximation. The Kelvin waves are studied and, for long wavelengths, the dispersion relation is obtained by an asymptotic matching method and compared with the non relativistic result.

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1 Introduction

An interesting issue in the study of relativistic vortices involves the similarities and differences between global strings and relativistic superfluid vortices [1, 2, 3, 4, 5].

This paper deals with a relativistic treatment of the almost ideal Bose gas at zero temperature and its approximation as a relativistic fluid. More specifically we study a cylindrical vortex and its Kelvin waves.

We know that an almost ideal Bose gas at zero temperature is well described by the Gross-Pitaevskii equation [6, 7, 8] which governs the evolution of the macroscopic wave function caracterising the condensate. This suggests (Sec.2) to start with a quantum field whose potential $m^2\phi^*\phi+\lambda(\phi^*\phi)^2$ does not exhibit spontaneous symmetry breaking, in order to represent weakly interacting relativistic Bose particles. By the Bogoliubov procedure we show that the classical field equation can be interpreted as a "relativistic Gross-Pitaevskii equation". In order to apply the Bogoliubov procedure we have to verify that the mode,

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in which the bosons condense, realizes the minimum of energy with the constraint that the conserved charge is fixed by the number of particles. This is done in Sec.3.

This preliminary study makes things very similar to the non relativistic counterpart (cf. for example [9]) and allows us to interpret the classical field equation as representing the motion of the condensate of an almost ideal gas at zero temperature.

In Sec.4 we establish the approximation of the field equations by a perfect isentropic irrotational fluid in the formulation of Lichnerowicz [10]. In Sec.5 the cylindrical stationary solution(that is a vortex) is studied. We begin with the exact solution in the approximate fluid theory and compare with an asymptotic solution of the exact theory.

In Sec.6 we study the Kelvin waves. The oscillations of a rectilinear vortex were initially studied by Lord Kelvin [11] in the context of the classical fluid. In Bose condensate theory the oscillations of a non relativistic quantum vortex were first discussed by Pitaevskii [7] who found in the long wavelength limit the dispersion relation:

$$\omega = \frac{q^2}{2m} \ln \frac{1}{qr_0} \tag{1}$$

where r_0 is the core radius. This result is refined by Grant [12] who has found:

$$\omega = \frac{q^2}{2m} \{ \ln \frac{2}{qr_0} - \gamma - 0.115 \} \tag{2}$$

where γ is the Euler constant. Davis and Shellard [1] have shown that a spinning global string behaves like a vortex in a superfluid medium. The dispersion relation of the Kelvin modes of such a vortex was given by Ben-Ya'acov [13]. In the non relativistic limit, he obtains

$$\omega = \frac{q^2}{2m} \{ \ln \frac{1}{q\epsilon} - \gamma + 1.5 \} \tag{3}$$

where ϵ is a cutoff parameter which depends on the core radius parameter δ of the string by

$$\ln \frac{\epsilon}{\delta} = 1.615 - \ln 2. \tag{4}$$

This last formula is obtained for a vortex ring. Comparison of (2) and (3) yields exactly the same formula if $r_0 = \delta$.

In this paper we cannot follow the method of [13] which uses explicitly an equation of motion of the vortices based on the spontaneous breaking symmetry of the potential. We shall rather transpose the method of the matched-asymptotic-expansions of Roberts and Grant [14, 12] to the relativistic field equations. The result is

$$\omega = \sqrt{1 - v_0^2} \frac{q^2}{2m} \{ \ln \frac{2}{qr_0} - \gamma - 0.115 \} \tag{5}$$

where v_0 is the velocity at $r = r_0$ of the classical relativistic fluid associated with the condensed bosons. Finally in Sec.7 we discuss some of our results, in particular we compare (5) with the dispersion relation of the Kelvin modes of a global string.

In this article the signature of the metric is $(-+++)$ and $c = \hbar = 1$.

2 Classical field equation and Bogoliubov procedure

Let us consider a phenomenological system of electrically neutral bosons described by a complex scalar field. It is worth noting that a complex field (non Hermitian operator) can describe particles and antiparticles without electric charge (cf. Landau and Lifchitz [15]). In this case the conserved charge Q is associated with the difference between the numbers of particles and antiparticles.

The lagrangian density is:

$$\mathcal{L} = -\partial_\mu \phi^* \partial^\mu \phi - V(\phi^* \phi) \tag{6}$$

with potential

$$V(\phi^*\phi) = m^2 \phi^* \phi + \lambda(\phi^*\phi)^2, \tag{7}$$

The constant m^2 is positive and represents the squared mass of the particles, the coupling constant λ is also positive so that the interaction is repulsive. If λ is small we can describe an almost ideal relativistic gas of bosons at temperature $T = 0$ by the field

$$\hat{\psi} = \phi + \hat{\psi}' \tag{8}$$

where ϕ is a classical field which will represent the condensate; it verifies the Euler-Lagrange equation

$$-\partial_\mu \partial^\mu \phi + m^2 \phi + 2\lambda \phi^* \phi \phi = 0 \tag{9}$$

and $\hat{\psi}'$ is a fluctuation.

If we suppose that the system is enclosed in a box of volume V, the field can be expanded in a Fourier series

$$\hat{\psi}(x) = \sum_{\mathbf{p}} \frac{1}{\sqrt{V}\sqrt{2\epsilon}} (a_{\mathbf{p}} e^{ipx} + b^+_{\mathbf{p}} e^{-ipx}), \tag{10}$$

with

$$px = -p^0 t + \mathbf{p} \cdot \mathbf{x}$$

and

$$\epsilon = p^0 = \sqrt{\mathbf{p}^2 + \epsilon_0^2}, \tag{11}$$

where ϵ_0 is a constant whose precise value will be fixed below. The operators $a_{\mathbf{p}}, b_{\mathbf{p}}$ and their conjugates $a^+_{\mathbf{p}}, b^+_{\mathbf{p}}$ are the annihilation and creation operators for the particles a and antiparticles b:

$$[a_{\mathbf{p}}, a^+_{\mathbf{p}}] = 1, \ [b_{\mathbf{p}}, b^+_{\mathbf{p}}] = 1. \tag{12}$$

Let us note that the Fourier expansion (10) and the commutation relations (12) usually given for free fields are valid also for interacting fields (cf. Lee [16]) . Of course in this case $a_{\mathbf{p}}$ and $b_{\mathbf{p}}$ and conjugates can have a complicated time dependence in contrast to free

fields where these operators are time independant if we choose for ϵ_0 the mass m of the free particle.

For a uniform weakly non ideal gas in a large volume V at temperature $T = 0$, almost all the N particles, $N_0 \approx N$, condense in the zero mode of the particles. As it will be justified in the following section, this mode represents a minimum of energy when we impose the constraint that the conserved charge Q is fixed: $Q = N_0$. Hence, since N_0 is a macroscopic number, following the Bogoliubov approximation [17], we can negect the commutator $[a_{\mathbf{0}}, a_{\mathbf{0}}^+] = 1$ as compared with the huge eigenvalue N_0 of $a_{\mathbf{0}}^+ a_{\mathbf{0}}$, so the operators $a_{\mathbf{0}}$, $a_{\mathbf{0}}^+$ can be approximated by classical numbers

$$a_{\mathbf{0}} \approx a_{\mathbf{0}}^+ \approx \sqrt{N_0}. \tag{13}$$

Therefore the Fourier expansion (10) can be decomposed

$$\hat{\psi} = \frac{\sqrt{N_0}}{\sqrt{V}\sqrt{2\epsilon_0}} e^{-i\epsilon_0 t} + \hat{\psi}' \tag{14}$$

where the classical field

$$\phi_0 = \frac{\sqrt{N_0}}{\sqrt{V}\sqrt{2\epsilon_0}} e^{-i\epsilon_0 t} \tag{15}$$

represents the condensate of the particles of zero momentum and the operator $\hat{\psi}'$ referring to the remaining non condensed particles is considered as a small perturbation.

Let us note that we have not fixed the arbitrary constant ϵ_0. Since ϕ_0 given by (15) must be a classical solution of the equation (9), ϵ_0, supposed positive, is fixed by

$$n_0 = \frac{\epsilon_0(\epsilon_0^2 - m^2)}{\lambda} \tag{16}$$

where

$$n_0 = \frac{N_0}{V} \tag{17}$$

is the constant density.

Let us remark that we consider a gas of bosons at low energy. More precisely it is a relativistic gas but a weakly relativistic gas where creation and destruction of antibosons are negligible. Of course we could also consider the symmetrical situation of a weakly relativistic gas of antibosons living in an antimatter world. In this case the constraint is $Q = -N_0$ where N_0 is the number of antibosons, the condensation would be on the zero mode of the antiparticles and the classical field representing the antiparticle condensate would be given by (15) with opposite phase.

The generalisation of the Bogoliubov approximation to the case of non uniform and time dependent configurations is given by (8) where the classical field ϕ, which represents the condensate, verifies the equation (9). This completes the comparison with the non relativistic theory (cf. for example [9]).

3 Minimum of energy at fixed conserved charge

In this section we show that the homogeneous solution (15) and (16) realises the minimum of energy when the number of particles is fixed.

It will be convenient to write the classical complex field $\phi(x)$ in polar form:

$$\phi(x) = F(x)e^{iS(x)}. \tag{18}$$

The Lagrangian (6) becomes

$$\mathcal{L} = -\partial_\mu F \partial^\mu F - F^2 \partial_\mu S \partial^\mu S - V(F^2) \tag{19}$$

and the Euler-Lagrange equations take the form

$$-\frac{\nabla_\mu \nabla^\mu F}{F} + \partial_\mu S \partial^\mu S + V'(F^2) = 0 \tag{20}$$

where $V'(F^2)$ is the derivative $\frac{dV}{dF^2}$,

$$\nabla_\mu (F^2 \partial^\mu S) = 0. \tag{21}$$

The 4-current and the energy density are given respectively by

$$j_\mu = -i(\phi^* \partial_\mu \phi - \partial_\mu \phi^* \phi) = 2F^2 \partial_\mu S \tag{22}$$

and

$$\mathcal{H} = (\partial_0 F)^2 + F^2(\partial_0 S)^2 + (\nabla F)^2 + F^2(\nabla S)^2 + V(F^2). \tag{23}$$

Let us introduce the conjugate momenta of the fields

$$\pi_F = \frac{\partial \mathcal{L}}{\partial F_{,0}} = 2\partial_0 F \tag{24}$$

and

$$\pi_S = \frac{\partial \mathcal{L}}{\partial S_{,0}} = 2F^2 \partial_0 S. \tag{25}$$

The charge and the Hamiltonian can be respectively written

$$Q = \int_V j^0 d^3x = \int_V -2F^2 \partial_0 S d^3x = \int_V -\pi_S d^3x \tag{26}$$

and

$$H = \int_V (\frac{1}{4}\pi_F^2 + \frac{1}{4}\frac{\pi_S^2}{F^2} + (\nabla F)^2 + F^2(\nabla S)^2 + V(F^2))d^3x. \tag{27}$$

The extremum of the energy H, when the conserved charge Q is fixed by the constant N_0, is determined by the variation

$$\delta\left(H - \mu(Q - N_0)\right) = 0 \tag{28}$$

where μ is a Lagrange multiplier. The variation on π_F, π_S, F, S, gives respectively

$$\pi_F = 0, \tag{29}$$

$$\frac{1}{2}\frac{\pi_S}{F^2} + \mu = 0, \tag{30}$$

that is

$$F = F(\mathbf{x}), \tag{31}$$

$$S = -\mu t + S_0(\mathbf{x}), \tag{32}$$

and implies that these functions are solutions of the equations (20), (21).

Bekenstein and Guendelman [18] show that this type of solution is a local energy minimum within its sector of fixed charge. In particular the homogeneous solution (15) and (16) belongs to this class of solutions with $\mu = \epsilon_0$ and realises the minimum of energy when the conserved charge is fixed by the number N_0 of particles, that is

$$H = \frac{3}{4}N_0\epsilon_0 + \frac{1}{4}N_0\frac{m^2}{\epsilon_0}. \tag{33}$$

We can show that (33) is effectively the energy minimum as follows. A solution of (28) can be written as a perturbation of the homogeneous solution (15), (16):

$$F = \sqrt{\frac{n_0}{2\epsilon_0}} + f(\mathbf{x}), \tag{34}$$

$$S = -\epsilon_0 t + S_0(\mathbf{x}) \tag{35}$$

where $f(\mathbf{x})$ is constrained by

$$\int_V \left(2\sqrt{\frac{n_0}{2\epsilon_0}}f(\mathbf{x}) + f^2(\mathbf{x})\right) d^3x = 0 \tag{36}$$

in order to maintain $Q = N_0$.

If we introduce (34) and (35) in (27), several terms are canceled by using (36) and we obtain finally

$$\begin{aligned} H = {} & \frac{3}{4}N_0\epsilon_0 + \frac{1}{4}N_0\frac{m^2}{\epsilon_0} \\ & + \int_V \left((\nabla f(\mathbf{x}))^2 + (\sqrt{\frac{n_0}{2\epsilon_0}} + f(\mathbf{x}))^2(\nabla S_0(\mathbf{x}))^2 + \lambda\left(2\sqrt{\frac{n_0}{2\epsilon_0}}f(\mathbf{x}) + f^2(\mathbf{x})\right)^2\right) d^3x \end{aligned} \tag{37}$$

The integral on the right hand side is obviously positive therefore (33) is the minimum with $Q = N_0$.

4 Fluid description

We shall express the modulus $F(x)$ of the field in the form

$$F^2(x) = \frac{n(x)}{2h(x)}. \tag{38}$$

Below it will turn out that $n(x)$ is the proper density of the particles and $h(x)$ the enthalpy per particle in the fluid approximation. The expression (38) can be suggested by the homogeneous solution (15) where $\frac{N_0}{V} = n_0$ is the constant density and ϵ_0 an energy scale fixed by the relation (16).

The 4-gradient of the phase $\partial_\mu S$ can be written as a covector C_μ in terms of a unit 4-vector u^μ:

$$C_\mu = \partial_\mu S = h u_\mu \tag{39}$$

hence from (38) the 4-current (22) takes the hydrodynamics form:

$$j_\mu = n u_\mu \tag{40}$$

where we interpret n (if it is positive) as the proper density and u^μ as the velocity 4-vector of the fluid.

From (39) we deduce also

$$\partial_\mu C_\nu - \partial_\nu C_\mu = 0 \tag{41}$$

and

$$C_\mu C^\mu = -h^2 \tag{42}$$

Equation (21) is the familiar continuity equation

$$\nabla_\mu j^\mu = 0. \tag{43}$$

In equation (20) the term $\frac{\nabla_\mu \nabla^\mu F}{F}$ can be neglected if F varies slowly compared to S. This term is analogous to the "quantum pressure" $\frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{n}}{\sqrt{n}}$ in the non relativistic theory [9].

Let us look at this approximation more closely. The equation (20) takes the form

$$\partial_\mu S \partial^\mu S + V'(F^2) = 0, \tag{44}$$

hence we have

$$-C^\mu C_\mu = m^2 + \lambda \frac{n}{h} \tag{45}$$

which combined with (42) gives

$$n = \frac{h(h^2 - m^2)}{\lambda}. \tag{46}$$

This relation $n(h)$ is identical with the relation $n_0(\epsilon_0)$ given by (16), but (46) is valid within the approximation considered here.

The equations (41),(42),(43) and

$$j_\mu = \frac{n(h)}{h} C_\mu \tag{47}$$

are the relativistic equations of a perfect isentropic irrotational fluid (cf. [10] and [19] where these equations appear in the same notations). For the most general framework in relativistic hydrodynamics (multicomponent fluids, superfluids), the reader is refered to [20] and [21].)

The nature of h is now clear. It is the enthalpy per particle (with the mass energy m included). The equation (46) is the equation of state which is automatically imposed.

If the quantities ρ and p designate respectively the proper energy density and the pressure, the enthalpy per particle can be written

$$h = \frac{\rho + p}{n}. \tag{48}$$

Since the entropy is zero we have

$$dp = ndh \tag{49}$$

The equations (48) and (49) define p and ρ in function of h and n. (49) combined with (46) can be integrated:

$$p = \frac{(h^2 - m^2)^2}{4\lambda} = \lambda F^4(x). \tag{50}$$

The velocity of sound $v_s = \sqrt{dp/d\rho}$ is given by

$$v_s^2 = \frac{n(h)}{hn'(h)} = \frac{h^2 - m^2}{3h^2 - m^2} < \frac{1}{3}. \tag{51}$$

We remark that $v_s^2 \to 1/3$ when $m \to 0$ which is the the correct ultrarelativistic limit.

The expressions(48) and (50) can also be obtained by identifying the energy-momentum tensor of (19) (in which the derivatives $\partial_\mu F$ are neglected), that is

$$T_{\mu\nu} = 2F^2 \partial_\mu S \partial_\nu S + \eta_{\mu\nu}(-F^2 \partial_\rho S \partial^\rho S - V(F^2)) \tag{52}$$

with the perfect fluid energy-momentum tensor. Finally we notice that the approximate equations can be obtained from a generic Lagrangian proposed by Carter [22] to describe the irrotational fluid in a condensate:

$$\mathcal{L} = -\frac{1}{2} F^2 \partial_\mu S \partial^\mu S - \mathcal{U}(F). \tag{53}$$

In our case:

$$\mathcal{U}(F) = \tfrac{1}{2}(m^2 F^2 + \lambda F^4). \tag{54}$$

5 Cylindrical vortex

5.1 Vortex solution of the fluid approximation

It is convenient to begin with the approximate theory governed by the equations (41), (42), (43) and (47) with the equation of state of the fluid (46).

We know (cf. [19], [23], [24]) that in cylindrical coordinates

$$ds^2 = -dt^2 + dz^2 + dr^2 + r^2 d\theta^2 \tag{55}$$

a stationary cylindrical solution is given by

$$C_\mu = (-E, L, 0, M) \tag{56}$$

where E, L, M are the three constants associated with the three symmetries. By a Lorentz transformation along z we can always choose $L = 0$ and have a circular flow.

The enthalpy h is a function of r given by

$$h^2 = -C^\mu C_\mu = E^2 - L^2 - \frac{M^2}{r^2}. \tag{57}$$

The equation of state of the fluid (46) yields two values $h = 0$ and $h = m$ for which n vanishes. When we move from the exterior toward the center of the vortex, $h(r)$ is decreasing; h is always larger than m, so n vanishes first at $h = m$ which determines the radius

$$r_0 = \frac{M}{\sqrt{E^2 - L^2 - m^2}}$$

of the core of the vortex. Below this value n becomes negative, which is meaningless in this approximation of classical fluid. Let us observe that with a familiar polytropic fluid the core is empty, n is zero. The sound velocity vanishes also on the radius r_0.

We shall now suppose for simplicity that we have a circular flow with $L = 0$. Using (46), (56), (57), we write the wave function of the vortex of the approximate equations:

$$\phi = \left(\frac{E^2 - \frac{M^2}{r^2} - m^2}{2\lambda} \right)^{1/2} e^{i(-Et+M\theta)} \tag{58}$$

where we have suppressed an undetermined constant phase.

To make this wave function single-valued, M must be an integer, noted ν. The domain of validity of (58) is $r \geq r_0$ where

$$r_0 = \frac{1}{\sqrt{E^2 - m^2}} \tag{59}$$

is the core radius for $\nu = 1$. The length r_0 sets the limit below which we cannot neglect the effect of the quantum term.

It is immediate to write the velocity of the circular flow

$$v = \frac{\nu}{Er}.$$

Let v_0 be this velocity at $r = r_0$ for $\nu = 1$, we have:

$$E = \frac{m}{\sqrt{1 - v_0^2}}. \tag{60}$$

5.2 Vortex solution of the exact theory

A vortex solution of the exact equations (20) and (21) must verify (41), (42), (43) and (47) whose stationary cylindrical solution is always given by (56) and (57). The relation between n and h is no longer (46) but n is a function of r and $F^2 = \frac{n(r)}{2h(r)}$ is also a function of r. So the vortex $\phi = F \exp iS$ is determined (with $L = 0$) by

$$S = -Et + \nu\theta \tag{61}$$

and by the the differential equation for $F(r)$ deduced from (20):

$$\frac{1}{r}\partial_r(r\partial_r F(r)) - \frac{\nu^2}{r^2}F(r) + (E^2 - m^2)F(r) - 2\lambda F^3(r) = 0 \tag{62}$$

This type of equation has been discussed in the literature many times (see for example [8]). Analytical solutions are not kown, but it is easy to discuss its asymptotic properties for $r \to 0$ and $r \to \infty$ and to compare with the solution of the fluid approximation. The presence of the term $\nu^2 F/r^2$ means that $F(r)$ must tend to zero when $r \to 0$ if the energy in the volume V has to be finite. For small r the cubic term $2\lambda F^3$ is the smallest and can be neglected, the solution is the Bessel function

$$F(r) = AJ_\nu[(E^2 - m^2)^{1/2} r] \tag{63}$$

as $r \to 0$.

As $r \to \infty$ the first two terms of (62) cancel:

$$F(r) = F_0 = \sqrt{\frac{E^2 - m^2}{2\lambda}}. \tag{64}$$

An approximation to the solution is obtained by matching the functions (63) and (64) and their derivatives. Let $r = a_0$ be the radius of the junction. The constant A is fixed by the continuity of F at $r = a_0$ (we are not concerned by this value) and a_0 by the continuity of the derivative $\frac{dF}{dr}(a_0)$:

$$J'_\nu[(E^2 - m^2)^{1/2} a_0] = 0. \tag{65}$$

So a_0 is given in term of the first zero of J'_ν, let say z_ν. For the winding number $\nu = 1$, $z_1 = 1.84$. The radius a_0 of the junction, which is naturally interpreted as the radius of the vortex, is given by

$$a_0 = \frac{1.84}{\sqrt{E^2 - m^2}}. \tag{66}$$

We observe that r_0 the radius of the core of the vortex in the fluid approximation and a_0 differ only in a numerical factor close to unity, they are typically of the same order of magnitude. The length scale $a_0 \sim r_0$ is of the order of the so called "healing length" [9]. This result indicates that the fluid approximation is a good practical model for the study of the vortices in a relativistic Bose gas condensate.

6 Kelvin waves on the cylindrical vortex

We consider now a generic perturbation of the vortex $\phi = F \exp(iS)$ determined by (61) and (62):

$$\tilde{\phi} = \phi + \Delta\phi \simeq e^{iS}(F + \Delta F + iF\Delta S). \tag{67}$$

Substituting (67) in the Euler-Lagrange equations and assuming that the perturbation is small, we can linearise in ΔF andin $F\Delta S$. Then it is possible to expand ΔF and $F\Delta S$ in wave modes and to consider each mode separately:

$$\Delta F = M(r)\cos(-\omega t + qz + \bar{\nu}\theta), \tag{68}$$

$$F\Delta S = N(r)\sin(-\omega t + qz + \bar{\nu}\theta). \tag{69}$$

$M(r)$ and $N(r)$ verify the linear equations

$$M'' + \frac{1}{r}M' + (E^2 + \omega^2 - q^2 - 6\lambda F^2 - m^2)M - \frac{1}{r^2}[(\bar{\nu}^2 + \nu^2)M + 2\bar{\nu}\nu N] + 2E\omega N = 0, \tag{70}$$

$$N'' + \frac{1}{r}N' + (E^2 + \omega^2 - q^2 - 2\lambda F^2 - m^2)N - \frac{1}{r^2}[(\bar{\nu}^2 + \nu^2)N + 2\bar{\nu}\nu M] + 2E\omega M = 0, \tag{71}$$

We can also [13] introduce equivalently

$$\Delta F + iF\Delta S = H_+(r)\exp i(-\omega t + qz + \bar{\nu}\theta) + H_-(r)\exp -i(-\omega t + qz + \bar{\nu}\theta) \tag{72}$$

with

$$H_+(r) = \frac{M(r) + N(r)}{2} \quad , \quad H_-(r) = \frac{M(r) - N(r)}{2} \tag{73}$$

which verify the linear equations:

$$\frac{1}{r}(rH'_\pm)' + [(E \pm \omega)^2 - q^2 - \frac{(\nu \pm \bar{\nu})^2}{r^2}]H_\pm = (m^2 + 4\lambda F^2)H_\pm + 2\lambda F^2 H_\mp \tag{74}$$

Since we are interested in the small motion of the central line of the vortex, let us concetrate in the neighbourhood of the origine. For small r, $F(r)$ is given by (63) hence

$$F(r) \sim Ar^{\nu}. \tag{75}$$

We can neglect the two last termes of the r.h.s. of Eqs.(74) which are approximed by Bessel equations whose the solutions are

$$H_{\pm} = BJ_{|\nu\pm\bar{\nu}|}[((E\pm\omega)^2 - q^2 - m^2)^{\frac{1}{2}}r] \propto r^{|\nu\pm\bar{\nu}|} \tag{76}$$

The center of the perturbed vortex is determined by the zero of the solution $\tilde{\phi}$ and since $F(r)$ vanishes at the origin, Eq.(67) requires that at least one of the functions $H_{\pm}$ is different of zero at the origin. Suppose it is H_+, we must have

$$\nu + \bar{\nu} = 0 \tag{77}$$

and consequently $H_- \propto r^{2\nu}$ is negligible. Hence near the origin we get

$$\tilde{\phi} = \exp i(-Et + \nu\theta)\,(Ar^{\nu} + H_+(r)\exp i(-\omega t + qz + \bar{\nu}\theta))\,. \tag{78}$$

Let the constant $B = -a^{\nu}A$, eq. (78) becomes

$$\tilde{\phi} = A\exp i(-Et)\left(r^{\nu}e^{i\nu\theta} - a^{\nu}\exp i(-\omega t + qz)\right). \tag{79}$$

In the case $\nu = 1$ the cancellation of $\tilde{\phi}$ is obtained for

$$r = a \quad , \quad \theta = -\omega t + qz \tag{80}$$

which is the helical motion of the center of the vortex, that is the Kelvin waves.

Let us return to the equations (70) and (71). If $E = 0$ and if we change m^2 into $-m^2$, they are identical to the corresponding equations for the global strings of [3]. We can verify that for $\nu = 1$, $\bar{\nu} = -1$, $\omega = 0$ and $q = 0$ there is an exact solution:

$$M(r) = F'(r) \quad , \quad N(r) = \frac{F(r)}{r}. \tag{81}$$

In the following we shall have always $\nu = -\bar{\nu} = 1$ since we are interested in Kelvin waves and we shall use non dimensional variables.

Let

$$y = \frac{r}{r_0}, \tag{82}$$

$$F(r) = F_0 f(\frac{r}{r_0}) = F_0 f(y) \tag{83}$$

where r_0 and F_0 are given respectively by (59) and (64).

The equation (62) becomes the standard equation

$$f'' + \frac{1}{y}f' + (1 - \frac{1}{y^2})f - f^3 = 0. \tag{84}$$

For large y the solution is

$$f(y) = 1 - \frac{1}{2y^2} - \frac{9}{8y^4} - \frac{161}{16y^6} + \cdots, \tag{85}$$

for small y we have

$$f(y) = ky + O(y^3) \tag{86}$$

where k is a constant which can be obtained numerically.

From the equations (68) and (69) we have naturally

$$M(r) = F_0 \bar{M}(y) \quad and \quad N(r) = F_0 \bar{N}(y). \tag{87}$$

We introduce also $\bar{\omega} = \omega r_0$, $\bar{q} = q r_0$, $\bar{E} = E r_0$, $\bar{m} = m r_0$, $t = r_0 \bar{t}$, $z = r_0 \bar{z}$. The equations (70) and (71) becomes:

$$\bar{M}'' + \frac{1}{y}\bar{M}' + (\bar{\omega}^2 - \bar{q}^2 + 1 - 3f^2(y))\bar{M} - \frac{2}{y^2}(\bar{M} - \bar{N}) + 2\bar{E}\bar{\omega}\bar{N} = 0, \tag{88}$$

$$\bar{N}'' + \frac{1}{y}\bar{N}' + (\bar{\omega}^2 - \bar{q}^2 + 1 - f^2(y))\bar{N} - \frac{2}{y^2}(\bar{N} - \bar{M}) + 2\bar{E}\bar{\omega}\bar{M} = 0 \tag{89}$$

Since $\nu = -\bar{\nu} = 1$ the solution of these equations can express the deplacement of the vortex. This is the case of the exact solution:

$$\bar{M}_0 = f'(y) \quad , \quad \bar{N}_0 = \frac{f(y)}{y} \tag{90}$$

obtained for $\bar{\omega} = \bar{q} = 0$.

We can now examine the solutions of Eqs. (88) and (89) and compute the frequency $\bar{\omega}$ in the limit $\bar{q} \to 0$ of the long wavelength oscillations. To solve the equations we shall follow the method of "matched asymptotic expansions" used by Roberts and Grant [14, 12].

We consider first the solution in the inner cylindrical region of radius y centred on the oz axis in the form of an expansion about the exact solution (90). This solution is then examined for $y \to \infty$. We consider also the solution in the outer region caracterized by a new stretched coordinate $s = \bar{q}y$ which is of order unity for large y. This solution is examined in the limit $s \to 0$. Finally the two asymptotic solutions ($y \to \infty$ and $s \to 0$) are matched in an overlap domain, where both the inner and outer expansion are valid.

From this matching we shall obtain the dispersion relation of the Kelvin waves.

6.1 Interior solution

We expand $\bar{M}$ and $\bar{N}$ around the exact solution $\bar{M}_0$ and $\bar{N}_0$:

$$\bar{M} = \bar{M}_0 + \bar{q}^2 \bar{M}_1 + \cdots \tag{91}$$

$$\bar{N} = \bar{N}_0 + \bar{q}^2 \bar{N}_1 + \cdots. \tag{92}$$

Then, substituting (91) and (92) into the equations (88) and (89) we obtain, by equating the coefficients of $\bar{q}^2$

$$\bar{M}_1'' + \frac{1}{y}\bar{M}_1' + \left(1 - \frac{2}{y^2} - 3f^2(y)\right)\bar{M}_1 + \frac{2}{y^2}\bar{N}_1 = f'(y) - 2\bar{E}\omega_1\frac{f(y)}{y}, \tag{93}$$

$$\bar{N}_1'' + \frac{1}{y}\bar{N}_1' + \left(1 - \frac{2}{y^2} - f^2(y)\right)\bar{N}_1 + \frac{2}{y^2}\bar{M}_1 = \frac{f(y)}{y} - 2\bar{E}\omega_1 f'(y), \tag{94}$$

with

$$\bar{\omega} = \omega_1\bar{q}^2. \tag{95}$$

To solve (93) and (94) we substitute $f(y)$ by the expansion (85) valid for large y.

First we search for solutions for the homogeneous system which is transformed in a linear equation of 4^{th} order in $\bar{M}_1$:

$$\begin{aligned} &\bar{M}_1^{(4)} + \frac{6}{y}\bar{M}_1^{(3)} + (-2 + \frac{7}{y^2} + \frac{8}{y^4} + \frac{76}{y^6} + \cdots)\bar{M}_1'' + (-\frac{10}{y} + \frac{1}{y^3} - \frac{16}{y^5} - \frac{380}{y^7} + \cdots)\bar{M}_1' \\ &+(-\frac{6}{y^2} - \frac{9}{y^4} - \frac{18}{y^6} + \cdots)\bar{M}_1 = 0 \end{aligned} \tag{96}$$

Infinity is an irregular singular point of rank 0 of this equation. The roots of the characteristic equation are $\pm\sqrt{2}$ and 0. The root 0 has a multiplicity 2. By applying standard techniques about asymptotic solutions of linear equations [25], it is possible to obtain asymptotic expansion of four independant solutions:

$$\begin{aligned} g_1 &= \tfrac{1}{y^3} + \tfrac{9}{2y^5} + \tfrac{483}{8y^7} + \cdots \\ g_2 &= \tfrac{1}{y} + \tfrac{2\ln y}{y^3} + \tfrac{101}{36y^3} + \tfrac{9\ln y}{y^5} + \tfrac{27}{4y^5} + \tfrac{483\ln y}{4y^7} + \cdots \\ g_3 &= \exp(-\sqrt{2}y)\tfrac{1}{\sqrt{y}}(1 - \tfrac{5}{8\sqrt{2}y} + \tfrac{65}{256y^2} - \tfrac{5981}{6144\sqrt{2}y^3} + \tfrac{181825}{393216y^4} + \cdots) \\ g_4 &= \exp(\sqrt{2}y)\tfrac{1}{\sqrt{y}}(1 + \tfrac{5}{8\sqrt{2}y} + \tfrac{65}{256y^2} + \tfrac{5981}{6144\sqrt{2}y^3} + \tfrac{181825}{393216y^4} + \cdots) \end{aligned} \tag{97}$$

From these solutions $\bar{M}_1$ we deduce the corresponding solutions $\bar{N}_1$:

$$\begin{aligned} f_1 &= \tfrac{1}{y} - \tfrac{1}{2y^3} - \tfrac{9}{8y^5} + \cdots \\ f_2 &= y + \tfrac{2\ln y}{y} + \tfrac{65}{36y} - \tfrac{\ln y}{y^3} - \tfrac{77}{18y^3} - \tfrac{9\ln y}{4y^5} + \cdots \\ f_3 &= \exp(-\sqrt{2}y)\tfrac{1}{\sqrt{y}}(-\tfrac{1}{y^2} + \tfrac{37}{8\sqrt{2}y^3} - \tfrac{2945}{256y^4} + \cdots) \\ f_4 &= \exp(\sqrt{2}y)\tfrac{1}{\sqrt{y}}(-\tfrac{1}{y^2} - \tfrac{37}{8\sqrt{2}y^3} - \tfrac{2945}{256y^4} + \cdots) \end{aligned} \tag{98}$$

From the solutions (97) and (98) we can construct the fundamental matrix of the homogeneous system. Then by the variation-of-constant formula, we obtain the solution of the inhomogeneous system (93) and (94). This solution depends on four arbitrary constants. By choosing two of them we can suppress in the solution the exponentials $\exp(\pm\sqrt{2}y)$ which are not compatible with the exterior solution. Finally, we obtain:

$$\bar{M}_1 = \frac{1}{2}\frac{\ln y}{y} + \frac{1}{y}(A_1 + \bar{E}\omega_1) + \cdots, \tag{99}$$

$$\bar{N}_1 = \frac{1}{2} y \ln y + A_1 y + \frac{1}{2y} \ln^2 y + \frac{\ln y}{y} \left(2A_1 + 2\bar{E}\omega_1 + \frac{3}{4} \right) + \frac{A_2}{y} + \cdots \tag{100}$$

where A_1 and A_2 are the two remaining constants which can be determined by matching the interior solution and the exterior solution.

By combining the equations (93) and (94) we can obtain an integral relation which will be useful later in determining ω. If we multiply (93) by $yf'(y)$ and (94) by $f(y)$ and add the two resulting expressions, using Eq. (84) we obtain:

$$\frac{d}{dy} \left(\frac{1}{y} f \bar{N}_1 + f \bar{N}_1' - f' \bar{N}_1 + y f' \bar{M}_1' - y f^3 \bar{M}_1 + y f \bar{M}_1 + f' \bar{M}_1 - \frac{1}{y} f \bar{M}_1 \right) =$$
$$-2\bar{E}\omega_1 (f^2)' + y (f')^2 + \frac{1}{y} f^2 \tag{101}$$

We integrate from 0 to ∞ and we obtain after substituting the value of $\bar{M}_1$, $\bar{N}_1$ and f for $y \to \infty$ and $y \to 0$:

$$2A_1 + \frac{1}{2} = -2\bar{E}\omega_1 + \left\{ \int_0^\infty y \left(f'\right)^2 dy + \lim_{A \to \infty} \left(\int_0^A \frac{1}{y} f^2 dy - \ln A \right) \right\} \tag{102}$$

In obtaining this relation, $\ln \infty$ has been canceled from the r.h.s. and the l.h.s. If $\bar{E} = 1$ the relation (102) is identical to the corresponding non relativistic relation of Grant [12]. The numerical value of the integral part is known:

$$\left\{ \int_0^\infty y \left(f'\right)^2 dy + \lim_{A \to \infty} \left(\int_0^A \frac{1}{y} f^2 dy - \ln A \right) \right\} = -0.115 \tag{103}$$

6.2 Exterior solution

Let us return to the equations (88) and (89) and expand them for large y. Replacing y by $\frac{s}{\bar{q}}$ where s is a streched coordinate since $\bar{q}$ is small, the equations become:

$$\bar{q}^2 \frac{d^2 \bar{M}}{ds^2} + \frac{\bar{q}^2}{s} \frac{d\bar{M}}{ds} + \left[\omega_1^2 \bar{q}^4 - \bar{q}^2 + 1 - 3 \left(1 - \frac{\bar{q}^2}{s^2} - \frac{2\bar{q}^4}{s^4} \right) \right] \bar{M}$$
$$-2 \frac{\bar{q}^2}{s^2} \left(\bar{M} - \bar{N} \right) + 2\bar{E}\omega_1 \bar{q}^2 \bar{N} = 0, \tag{104}$$

$$\bar{q}^2 \frac{d^2 \bar{N}}{ds^2} + \frac{\bar{q}^2}{s} \frac{d\bar{N}}{ds} + \left[\omega_1^2 \bar{q}^4 - \bar{q}^2 + 1 - \left(1 - \frac{\bar{q}^2}{s^2} - \frac{2\bar{q}^4}{s^4} \right) \right] \bar{N}$$
$$-2 \frac{\bar{q}^2}{s^2} \left(\bar{N} - \bar{M} \right) + 2\bar{E}\omega_1 \bar{q}^2 \bar{M} = 0. \tag{105}$$

Let us expand $\bar{M}$ and $\bar{N}$ in series of $\bar{q}^2$:

$$\begin{aligned} \bar{M} &= M_{E1} + \bar{q}^2 M_{E2} + \cdots \\ \bar{N} &= N_{E1} + \bar{q}^2 N_{E2} + \bar{q}^4 N_{E3} + \cdots \end{aligned} \tag{106}$$

Substituting into (104) and equating successive powers of $\bar{q}^2$ we obtain for the powers zero and two of $\bar{q}$, respectively

$$M_{E1} = 0, \tag{107}$$

$$M_{E2} = \left(\frac{1}{s^2} + \omega_1 \bar{E}\right) N_{E1}. \tag{108}$$

Substituting (106) into (105) we find that the equality is identically satisfied at the zero order of $\bar{q}$. For the orders $\bar{q}^2$ and $\bar{q}^4$ we obtain, respectively:

$$N''_{E1} + \frac{1}{s} N'_{E1} - \left(1 + \frac{1}{s^2}\right) N_{E1} = 0, \tag{109}$$

$$N''_{E2} + \frac{1}{s} N'_{E2} - \left(1 + \frac{1}{s^2}\right) N_{E2} = -\left(\omega_1^2 + \frac{2}{s^4}\right) N_{E1} - \left(\frac{2}{s^2} + 2\bar{E}\omega_1\right) M_{E2}. \tag{110}$$

Let us note that the coefficient multiplying N_{E3} is zero.

The solution of (109) bounded at infinity is the modified Bessel function of the first order $K_1(s)$. Its expansion about $s = 0$ is:

$$N_{E1} = C\left\{\frac{1}{s} + \frac{s}{2}\ln\frac{s}{2} - \frac{s}{4}(1 - 2\gamma) + \frac{s^3}{16}\ln\frac{s}{2} - \frac{s^3}{32}\left(\frac{5}{2} - 2\gamma\right) + \cdots\right\} \tag{111}$$

where C is an arbitrary parameter and γ the Euler contant.

From (108) we have M_{E2}. Now the r.h.s. of Eq. (110) is known, since (111) is the solution of the homogeneous part of the equation (110), by the variation-of-constant formula we obtain the solution of (110):

$$\begin{aligned} N_{E2} = C\Bigg\{ & -\frac{1}{2s^3} + \frac{\ln^2(s/2)}{2s} + \frac{\ln s}{s}\left(\frac{1}{4} + \gamma + 2\bar{E}\omega_1\right) \\ & + \frac{1}{s}\left(\frac{9}{8} - \frac{\gamma}{4} + \frac{\ln 2}{4} - \frac{\ln^2(2)}{2}\right) + \cdots\Bigg\} \end{aligned} \tag{112}$$

From (106), (107), (108), (111), and (112), we have:

$$\bar{M} = \bar{q}^2 C\left\{\frac{1}{s^3} + \frac{1}{2s}\ln\frac{s}{2} + \frac{1}{s}\left(-\frac{1}{4} + \frac{\gamma}{2} + \bar{E}\omega_1\right) + s\ln\frac{s}{2}\left(\frac{1}{16} + \frac{\bar{E}\omega_1}{2}\right)\right\}, \tag{113}$$

$$\begin{aligned} \bar{N} = & \, C\left\{\frac{1}{s} + \frac{s}{2}\ln\frac{s}{2} - \frac{s}{4}(1 - 2\gamma) + \frac{s^3}{16}\ln\frac{s}{2} - \frac{s^3}{32}\left(\frac{5}{2} - 2\gamma\right) + \cdots\right\} \\ & + \bar{q}^2 C\Bigg\{-\frac{1}{2s^3} + \frac{1}{2s}\ln^2\frac{s}{2} + \frac{\ln s}{s}\left(\frac{1}{4} + \gamma + 2\bar{E}\omega_1\right) \\ & + \frac{1}{s}\left(\frac{9}{8} - \frac{\gamma}{4} + \frac{\ln 2}{4} - \frac{\ln^2(2)}{4}\right) + \cdots\Bigg\}. \end{aligned} \tag{114}$$

6.3 Matching

The interior solution for $y \to \infty$ must match the exterior one for $s \to 0$ in an overlap domain. For the comparison we express the inner solution in term of s:

$$\bar{M} = \bar{q}^3 \left\{ \frac{1}{s^3} + \frac{1}{2s} \ln \frac{s}{2} + \frac{1}{s} \left(\frac{\ln 2}{2} - \frac{1}{2} \ln \bar{q} + A_1 + \bar{E}\omega_1 \right) \right\} + \cdots, \tag{115}$$

$$\begin{aligned} \bar{N} &= \bar{q} \left\{ \frac{1}{s} + \frac{s}{2} \ln \frac{s}{2} + s \left(\frac{\ln 2}{2} - \frac{1}{2} \ln \bar{q} + A_1 \right) \right\} \\ &+ \bar{q}^3 \left\{ -\frac{1}{2s^3} + \frac{1}{2s} \ln^2 \frac{s}{2} + \frac{\ln s}{s} \left(\ln 2 - \ln \bar{q} + 2A_1 + 2\bar{E}\omega_1 + \frac{3}{4} \right) \right. \\ &+ \left. \frac{1}{s} \left(-\frac{\ln^2 2}{2} + \frac{\ln^2 \bar{q}}{2} - \ln \bar{q} \left(2A_1 + 2\bar{E}\omega_1 + \frac{3}{4} \right) + A_2 \right) + \cdots \right\}. \end{aligned} \tag{116}$$

A comparison of (113), (114) with (115), (116) for small s shows that

$$C = \bar{q} \quad , \quad 2A_1 + \frac{1}{2} = \gamma - \ln \frac{2}{\bar{q}}. \tag{117}$$

It is surprising that these two equalities are identical to the corresponding relations of Grant [12]. In fact, (117) is satisfied at the order $\bar{q}$ and $\bar{q}^3$ for the three first order of s in $\bar{M}$ and $\bar{N}$.

Then using (95), (102), (103) and (117) we obtain

$$\bar{E}\bar{\omega} = \frac{\bar{q}^2}{2} \left(\ln \frac{2}{\bar{q}} - \gamma - 0.115 \right). \tag{118}$$

If we return to the physical variables, we have the dispersion relation:

$$\omega = \frac{q^2}{2E} \left(\ln \frac{2}{qr_0} - \gamma - 0.115 \right), \tag{119}$$

that is, with (60):

$$\omega = \sqrt{1 - v_0^2} \frac{q^2}{2m} \left(\ln \frac{2}{qr_0} - \gamma - 0.115 \right), \tag{120}$$

where r_0 is the core radius (59) and v_0 the velocity at $r = r_0$ of the relativistic classical fluid.

7 Conclusion

In the begining of this paper we consider the relations between the quantum nature of the condensate of an almost ideal relativistic Bose gas and its phenomenological properties

as a relativistic superfluid. The quantum origin of the description imposes inevitably the equation of state of the fluid (46) which gives an acceptable velocity of sound. Incidently, we note that if we choose a potential with spontaneously broken symmetry (by changing m^2 into $-m^2$ in (7)) the velocity of sound would always be greater than $1/\sqrt{3}$.

The study of a cylindrical vortex has shown that this equation of state makes sense outside the core of the vortex whose the radius is approximately of the order of the healing length which indicates the onset of the quantum effects.

Concerning the Kelvin waves, it is surprising that the mathematical calculations are so close the non relativistic ones. Yet the field equation differs from the Gross-Pitaevskii equation essentially in the second time derivative. Consequently the main equations (88), (89) are different from the corresponding ones of [12]. The new dispersion formula (120) differs from the nonrelativistic one by the relativistic factor $\sqrt{1-v_0^2}$ where v_0 represents the velocity of the fluid in $r = r_0$ the radius of the core of the vortex.

It is interesting to compare with the spinning global strings which behave like vortices in a superfluid medium. Ben-Ya'acov [13] has given a covariant dispersion relation of the Kelvin waves. In the rest frame of the materiel medium, for an unit winding number and $\omega < q$, using (4), it can be express as

$$\omega = \frac{q^2(1-v_1^2)}{2E}\left(\ln\frac{2}{q\sqrt{1-v_1^2}\delta} - \gamma - 0.115\right), \tag{121}$$

where $v_1 = \frac{\omega}{q}$ is the velocity along z of the Lorentz frame in which the pertubed vortex is an helix at rest. E is not any more given by (60) but tends towards the particle mass in the non relativistic limit.

The formula (120) and (121) are different but, both are compatible at weak energy with the Grant formula (2). However it is worth stressing that (120) is certainly the better dispersion relation for a weakly relativistic almost ideal Bose gas. The superfluid medium concerned by the dispersion relation (121) is unusual as the velocity of sound is greater than $1/\sqrt{3}$

To conclude we note that relativity does not seem very useful in the laboratory experiments on the condensates but could be of interest in astrophysics.

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